Linearizing Computing the Power Set with OpenMP

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Introduction

- This paper presents 4 methods for computing the power set.
  - Three methods are serial.
  - One method is parallel.

- The three serial methods are:
  - Recursive algorithm.
  - Count-in-binary algorithm.
  - Disjunctive normal form algorithms --- serial version, no OpenMP directives.

- The one parallel method is:
  - Disjunctive normal form algorithms --- parallel version, uses OpenMP directives.

Methods - Overview

- Each method will be briefly explained next.

Methods – The Recursive Algorithm

- The recursive method contains two algorithms:
  1) One algorithm partitions the problem by incrementing two counters ***k*** and ***m*** in recursive calls.
  2) The other algorithm creates the actual sets in the power set $P(\prod)$ from a set of integers.
- When ***m*** > ***n*** = $|\prod|$, then the first algorithm terminates.
- Both algorithms create 2^n-1 sets in the power set
  - Excluding the empty set

Methods – The Count-in-Binary Algorithm

- The count-in-binary (CIB) method creates 2^n-1 sets in the power set.
- The method is simplistic:
  1) Count from 0 to 2^n-1 in decimal
  2) Convert the count to binary
  3) Include/exclude elements from the given set $\prod$ using the binary number
     1) 0 = exclude an element from $\prod$
     2) 1 = include an element from $\prod$
- This particular implementation of the CIB algorithm reverses the ordering of the elements in the sets

Methods – The Count-in-Binary Algorithm

- For example, instead of computing the set {a, b, c}; the CIB algorithm computes {c, b, a}.
  - Which are equivalent sets
- We make no effort to pad the binary numbers to the left with zeros.
- We make no effort to sort the computed sets so that the sets are more aesthetically pleasing.
- Hence, we present a Laissez-Faire CIB algorithm.

Methods – The Disjunctive Normal Form Algorithm

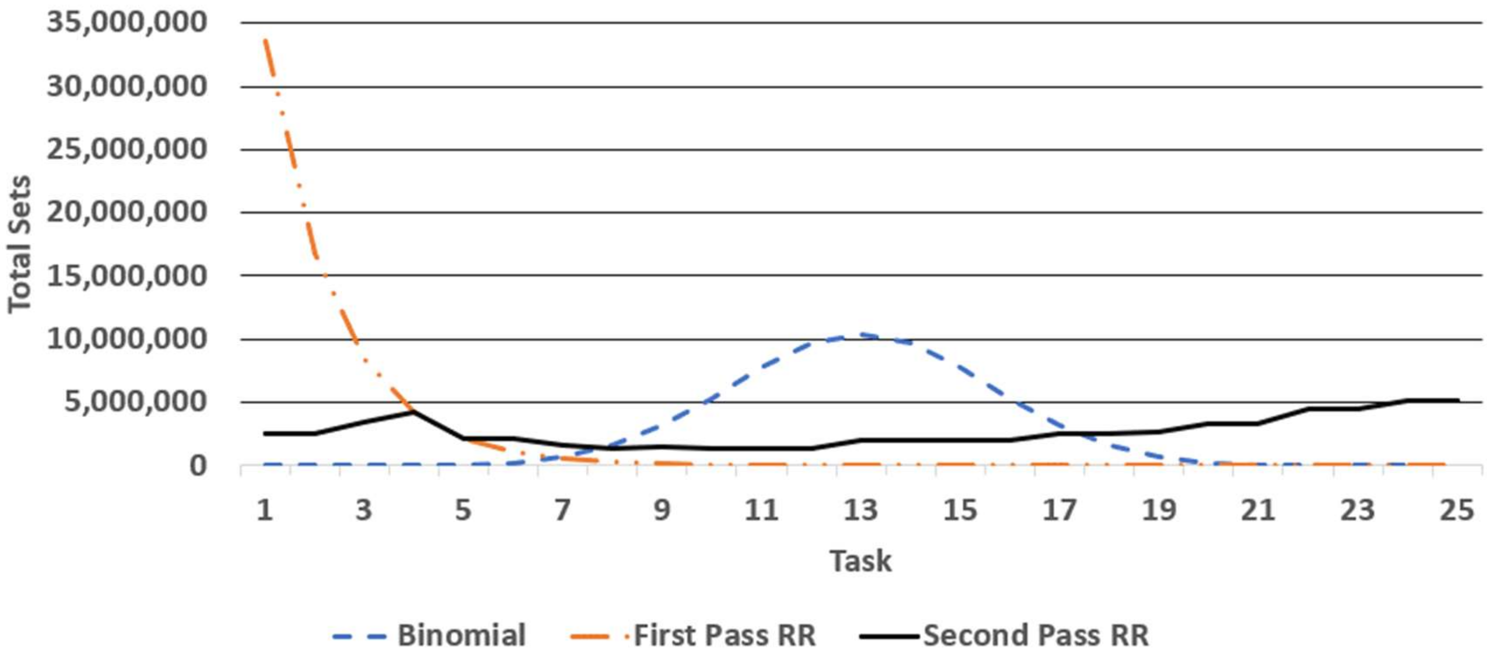
- The disjunctive normal form (DNF) method contains preprocessing steps.
  - The partition of the binomial coefficient function (BCF)
  - Load balancing using the partition of the BCF (PBCF)

- The BCF is as follow:

$$\binom{n}{m} = \frac{n!}{(n-m)!m!},\ 0\ \leq m \leq n.$$

- The BCF is a symmetric function.
  - It sums to $\boldsymbol{2^n}$ for $\boldsymbol{m = \{0, 1, \ldots, n\}}$.
  - The maximum is at $\lfloor \mathbf{n/2} \rfloor$ for even n.
  - See graph on next slide (blue dashed line).

Methods – The Disjunctive Normal Form Algorithm



Methods – The Disjunctive Normal Form Algorithm

- We partition the BCF "given" the first element in the power set.
  - This will be explained later.
- Note: Ignore sets with cardinality 1 or ***n***.

- Instead of programming the BCF function, we program the PBCF.

$$h(n, m|S = s) = \frac{\prod_{i=1}^{m-1}(n - m) - (s - 1) + i}{\prod_{i=1}^{m-1} i}$$

Methods – The Disjunctive Normal Form Algorithm

- The variable ***s*** is the cardinal number to the first element in the set.

- Two noticeable characteristics of the PBCF:
  - One-half of the values are zero
  - The function drops significantly for small ***s***

- Only certain arrangements in the powerset are allowed using the DNF algorithms.
  - Due to problem restrictions.

- The next slide gives an example.

Methods – The Disjunctive Normal Form Algorithm

- Suppose $\prod$ = {a, b, c, …, z} and ***m*** = 2.

- Consider the sets {a, b}, {a, c}, {a, z}, …, {x, y}, {x, z}, {y, z}.
  - The number of sets beginning with the element x is $h(n, m|S = 24)$ = 2.
  - The number of sets beginning with the element y is $h(n, m|S = 25)$ = 1.
  - The number of sets beginning with the element z is $h(n, m|S = 26)$ = 0.

- But,
  - The number of sets beginning with the element a is $h(n, m|S = 1)$ = 25.
  - The number of sets beginning with the element b is $h(n, m|S = 2)$ = 24.
  - The number of sets beginning with the element c is $h(n, m|S = 3)$ = 23.

Methods – The Disjunctive Normal Form Algorithm

- Remove the restriction ***m = 2***. Then, $2 \leq$ ***m*** ≤ 25.

- Only one additional set {x, y, z} is added to $h(n, m|S = 24)$.

- Many sets are added to $h(n, m|S = 1)$, $(n, m|S = 2)$, and
$h(n, m|S = 3)$.
    - $h(n, m|S = 1)$= 33,554,430
    - $h(n, m|S = 2)$= 16,777,215
    - $h(n, m|S = 3)$= 8,388,607

- This is due to the nature of the problem.
- With the cardinality and the partition, it is possible to break-up the BCF which leads to computing the power set faster in a parallel computing environment.

Methods – The Disjunctive Normal Form Algorithm

- Next, we discuss load balancing.

- We perform load balancing before the DNF algorithms are run.

- The load balancing algorithm is a 2-pass algorithm:
  - Pass 1 computes an (***n-1***) x (***n-1***) table using the PBCF. The columns represent the cardinality ***m***. The rows represent ***s***.

  - Pass 2 redistributes the ***q***-maximums to the rows in the table with the least number of sets (using the row totals).

Methods – The Disjunctive Normal Form Algorithm

- Load balancing ensures that a single task does not compute all of the sets with a cardinality close to ***m*** = $\lfloor$***n/2***$\rfloor$ and ***s*** equal to 1.

- Instead of arbitrarily setting ***q***, we calculate the ***q***-maximums in the (***n-1***) x (***n-1***) table using the following formula:

$$q_i = \begin{cases} 1, & \text{If } \sum_{m=1}^{n} h(n, m|S = s) < \max_{(i)}. \\ 0, & \text{Otherwise.} \end{cases}$$

where ***i*** is bounded by ***i*** $\in$ {1, 2, …, n-1}; $\max_{(i)}$ is the ***i***-th largest integer in the table; the sum $\sum_{i=1}^{n-1} q_i$ equals to the number of ***q***-maximums.

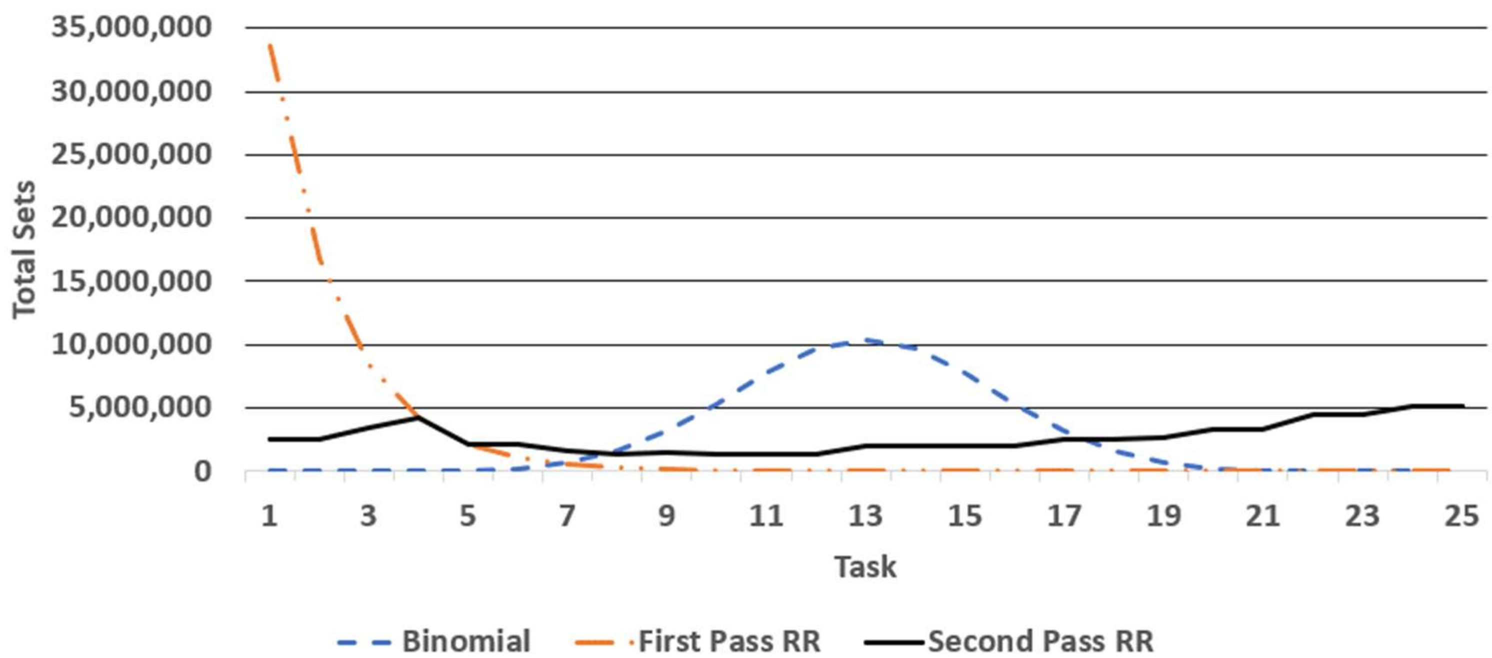
Methods – The Disjunctive Normal Form Algorithm

- For the trivial case ***m*** = 1, the 2-pass round robin algorithm simply puts the ***n*** computations into a single task.

- The figure on the next slide compares the round robin distribution to the BCF.
  - The flat line (in black) shows the final distribution after the 2-pass round robin algorithm.
  - We prefer the flat line compared to the other two distributions when computing the power set.

- The slide after next (slide 17) shows a partial (***n-1***) x (***n-1***) table with ***n*** = 12.
  - It is always the case that the upper, center part of the table needs to be load balanced.
  - ***q*** = 7
  - Do not forget to zero-out the ***q***-maximums.

Methods – The Disjunctive Normal Form Algorithm



Methods – The Disjunctive Normal Form Algorithm

m =	1	2	3	4	5	6	7	8	9	10	11	Row Totals
	12	11	55	165	**330**	**462**	**462**	**330**	165	55	11	2,058
	0	10	45	120	**210**	**252**	**210**	120	45	10	1	1,023
	0	9	36	84	126	126	84	36	9	1	0	511
	0	8	28	56	70	56	28	8	1	0	0	255
	0	7	21	35	35	21	7	1	0	0	0	127
	0	6	15	20	15	6	1	0	0	0	0	63
	0	5	10	10	5	1	0	0	0	0	0	31
	0	4	6	4	1	0	0	0	0	0	0	15
	0	3	3	1	0	0	0	0	0	0	0	7
	0	2	1	0	0	0	0	0	0	0	0	3
	0	1	0	0	0	0	0	0	0	0	0	1

Methods – The Disjunctive Normal Form Algorithm

- Some notes:
  - The most computer intensive sets to compute are those sets that have been redistributed by the round robin algorithm.
  - The round robin algorithm implements the ***q***-maximums by writing snippets of code which has to be inserted into the tasks.
  - Using both the PBCF and the 2-pass round robin algorithm, the entire power set for ***n*** = 26 can be computed in 55 seconds on the laptop used in [5], [6]. This is a 42.7% reduction in run-time.

Methods

- The differences between the CIB method and the DNF method:
  - The CIB method contains two algorithms. The DNF method contains ***m-1*** algorithms --- one algorithm for each cardinality ***m***, $\mathbf{1 \leq m \leq n\text{-}1}$.
  - The CIB algorithm terminates after $\mathbf{2^n-1}$ iterations. The DNF algorithms terminate after a pre-determined maximum has been reached.
  - The CIB main loop contains a single DO loop. The DNF algorithms' loops contain ***m*** loops for each cardinality.

Methods

- Advantages of each method:
  - The recursive algorithm is easy to program.
  - The CIB algorithm is easy to understand and easy to program.
  - The DNF algorithm runs in linear time in a parallel computing environment.

- Disadvantages of each method:
  - The recursive algorithm runs in exponential run-time.
  - The CIB algorithm runs in exponential run-time.
  - The DNF algorithm requires pre-processing.

Empirical Evaluation

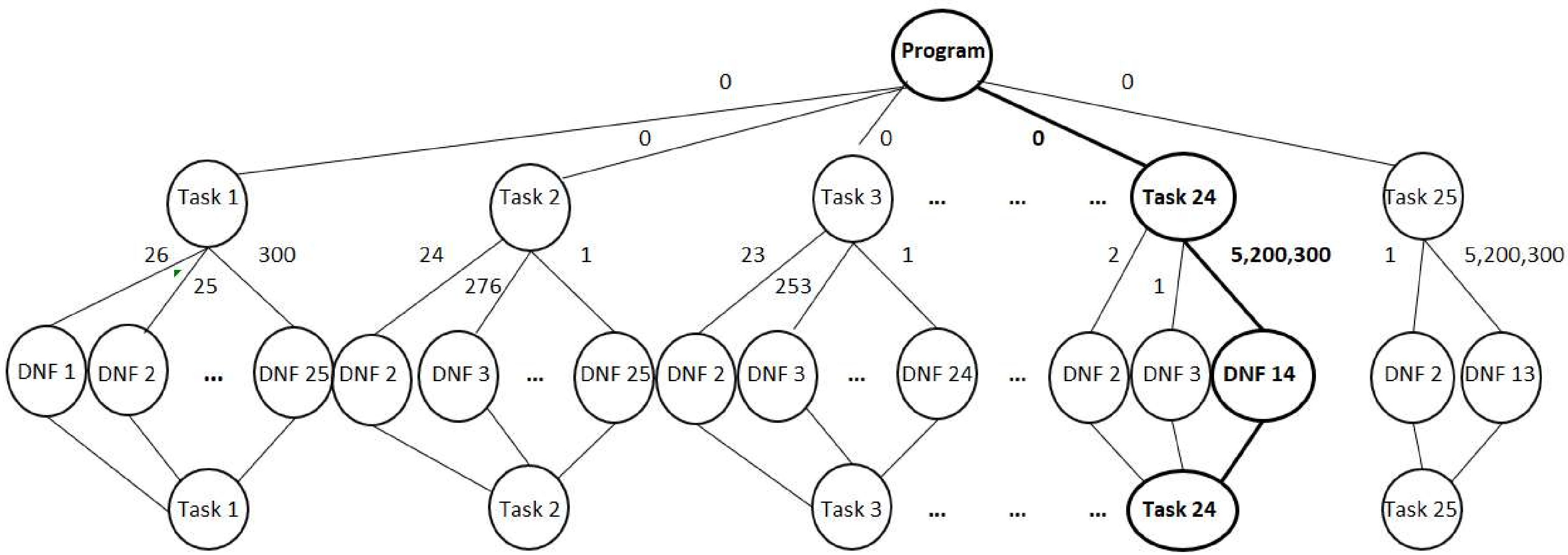
- We evaluate the methods on the Stampede2 supercomputer using the Skylake (SKX) compute nodes.

- We will:
  - Construct a task graph to show potential parallelism.
  - Run the Intel Advisor to show the top 5 time consuming loops.
  - Construct a scalability curve.
  - Summarize the results of the algorithms.
  - Outline a method to compute the power set for large ***n***.

Empirical Evaluation

- We construct a task graph of the DNF algorithm to show the *possible parallelism* in the program.
- The widest part of the graph shows the possible parallelism.
- The critical path shows maximum run-time of the program.
  - Because computing 5,200,300 sets with 14 nested loops is more computer intensive than computing 5,200,300 sets with 13 nested loops.
- See the Figure on the next slide.

Empirical Evaluation



Empirical Evaluation

- The Intel Advisor is a source code profiling tool.
- The Intel Advisor shows the top 5 time consuming loops.
  - The algorithms with the largest number of sets to compute and those algorithms with the greatest number of nested loops about the center $\boldsymbol{n}/2$.
  - Take the most time to compute the power set.
- See the Table on the next slide.

Empirical Evaluation

Top time-consuming loops			
Loop	Self-Time	Total Time	Trip Counts
[loop in dnf_new_14]	3.582 s	16.330 s	1
[loop in dnf_new_13]	3.511 s	16.360 s	1
[loop in dnf_new_12]	3.210 s	14.029 s	1
[loop in dnf_new_15]	2.779 s	13.960 s	1
[loop in dnf_new_11]	2.441 s	10.400 s	1

Empirical Evaluation

- The Intel Advisor Source Code Profiling tool is useful at times
- It can be used to confirm information you already know
- For instance, the top 5 time-consuming loops
  - You know which ones they are
  - The Intel Advisor tool confirms this

Empirical Evaluation

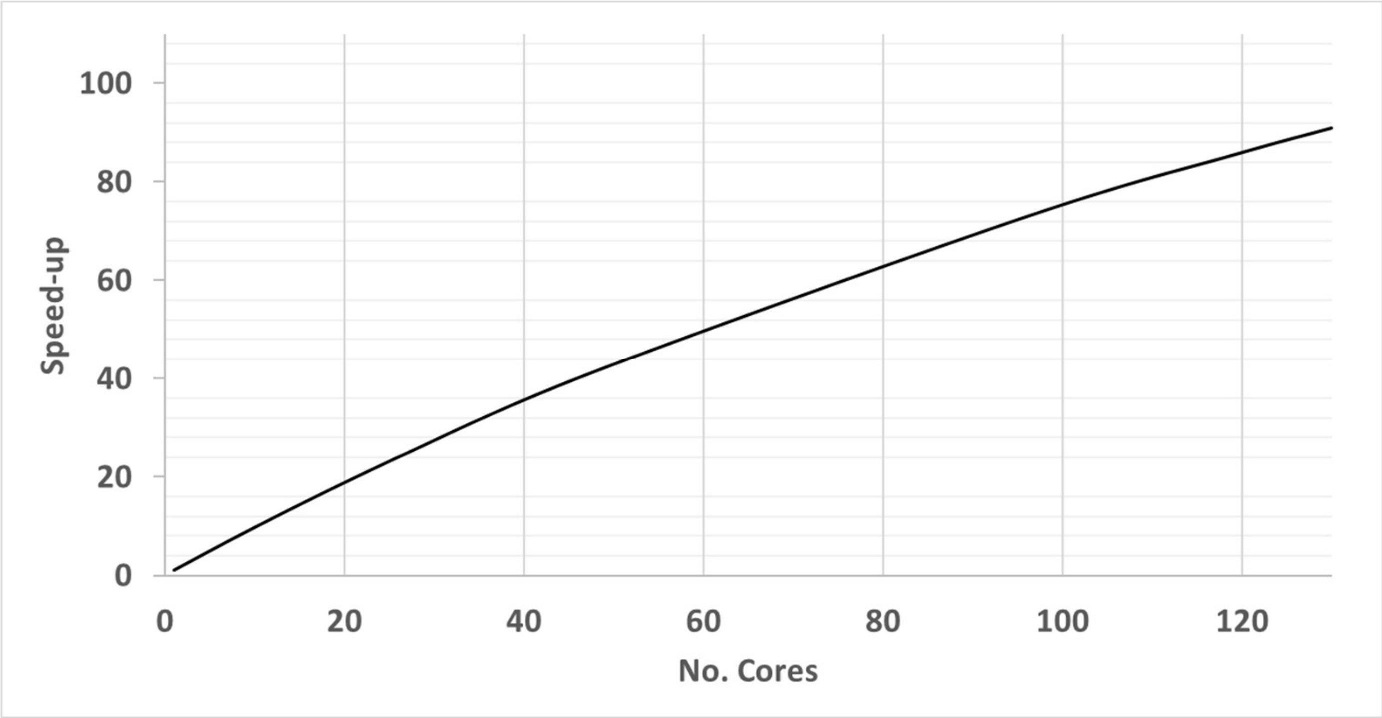
- Some notes on the Intel Advisor.

- The Intel Advisor suggests changing the data type inside the loop so that it matches
  - This will have a better chance of using the full vector register width
  - I modified the code to use the I(4) data type
  - The code ran twice as slow

Empirical Evaluation

- The next slide shows the scalability curve for the OpenMP DNF algorithm.
  - The scalability graph shows a linear relationship between cores versus speed-up.
- We estimated the percentage amount of serial code using Equation (6) in the paper.
- Then applied Amdahl's law to obtain the scalability curve.

Empirical Evaluation



Empirical Evaluation

- The following table summarizes the results of the timing study of the different methods.

n	T0	T1	T2	Tp	q	CV	Sp	Ep
15	0.04	0.04	0.2	0.2802	9	0.6	0.7	1.4%
16	0.1	0.1	0.2	0.1832	10	0.3	1.2	2.5%
17	0.2	0.2	0.3	0.2549	11	0.7	1.0	2.0%
18	0.4	0.3	0.3	0.2155	12	0.5	1.3	2.8%
19	0.8	0.7	0.3	0.1174	13	0.2	2.8	5.9%
20	1.6	1.5	0.4	0.2557	14	0.7	1.7	3.5%
21	3.3	3.1	0.6	0.1097	15	0.1	5.4	11.3%
22	7.1	6.5	1.0	0.1770	16	1.1	5.5	11.4%
23	14.5	13.5	1.8	0.1487	17	0.8	11.8	24.6%
24	30.6	28.0	3.4	0.1149	18	0.3	29.3	61.0%
25	62.7	58.2	6.7	0.1733	18	0.4	38.5	80.3%
26	131.4	117.4	13.6	0.2746	20	0.5	49.5	103.1%
Avg							12.4	25.8%
Min							0.7	1.4%
Max				0.280196			49.5	103.1%

Empirical Evaluation

- T_0 = recursive algorithm (seconds)
- T_1 = CIB algorithm (seconds)
- T_2 = non-OpenMP DNF algorithms (seconds)
- T_p = OpenMP DNF algorithms (seconds)

- Only one coefficient of variation (CV) above 1.0
  - Much variability about the average.

- We obtain an efficiency (EP) of 100% (accounting for variability) because the serial algorithm T_2 ran poorly and the parallel algorithm T_p ran efficiently.

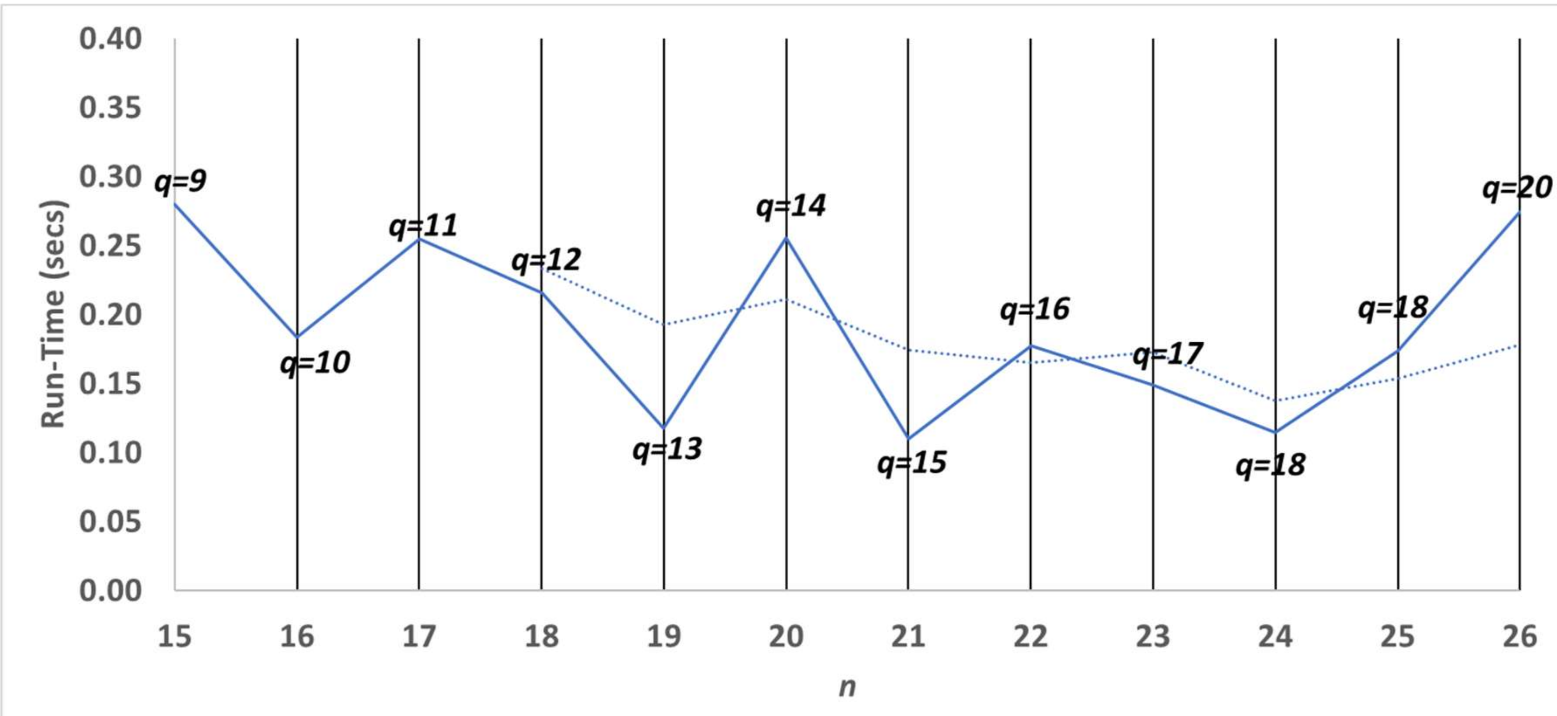
Empirical Evaluation

- The serial algorithms T_0, T_1, and T_2 have exponential run-time curves.

- The T_p parallel algorithm (OpenMP DNF algorithm) has a non-exponential run-time curve.

- The graph on the next slide shows the graph for the input sizes versus the run-times for the OpenMP DNF algorithm.

Empirical Evaluation



Empirical Evaluation

- Recommendations from the timing study
- The Stampede2 supercomputer is a shared machine
  - If another job is thrashing while your job is running, this will affect your timing study
  - Run your job numerous times on different days to get a good timing

Configuration Management

- We computed the power set in its entirety for n = 15, 16, …, 26.
- Consider computing the power set for n = 150 and n = 45,136.
- Obvious some implementation limitations will come up:
  - Integer exceeds machine limits.
  - Segmentation fault.
  - A single user can only have 25 jobs in queue at a time.

Configuration Management

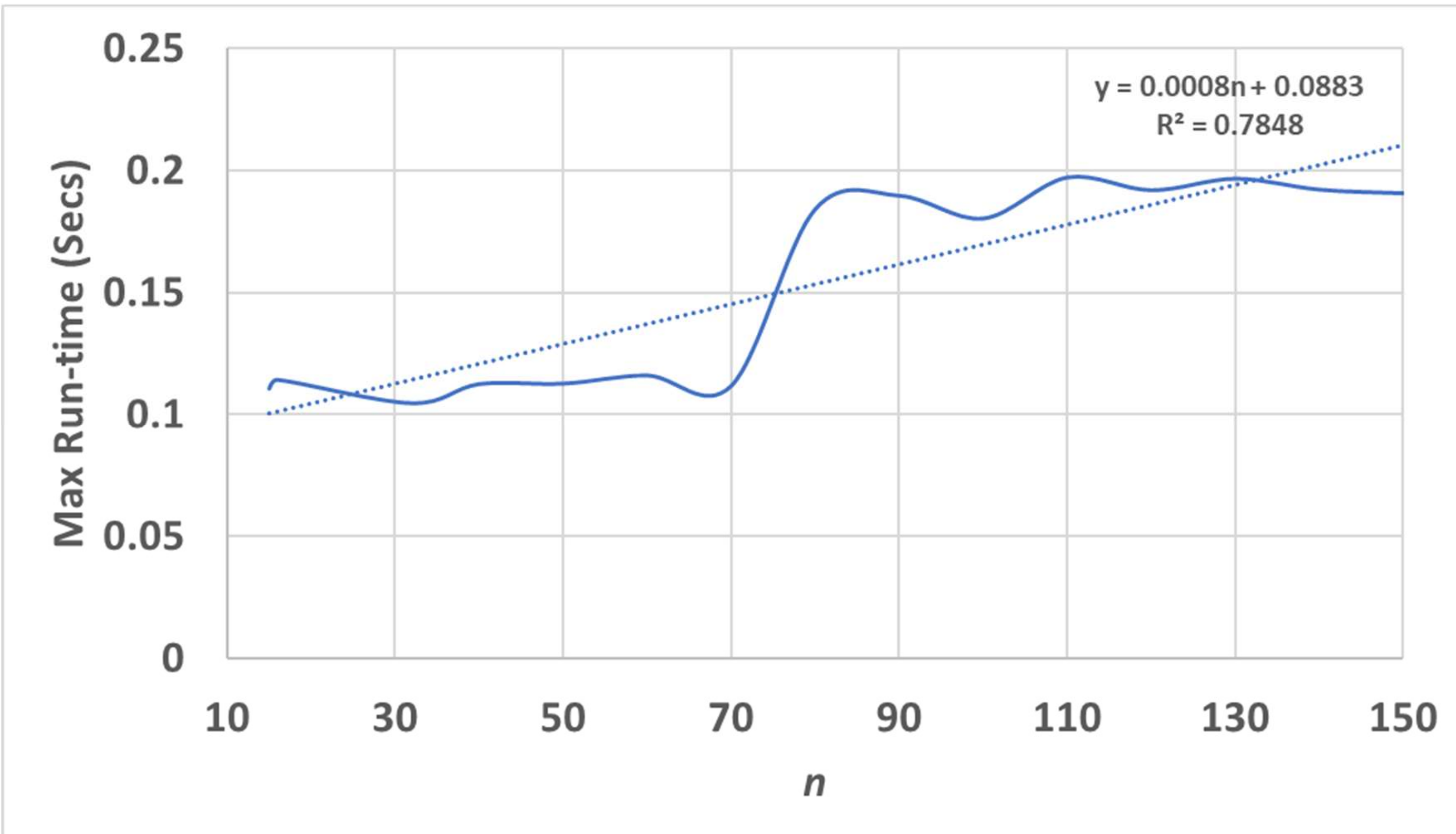
- Some possible workarounds include:
  - Use the –fno-range-check option when compiling the code.
  - Overwrite the values in the array when the index reaches $\boldsymbol{2^{31}}$.
  - Wait until some of the jobs have finished, then submit more jobs.

- The *q*-maximums are a second source of exponentiation.
  - These values must be partitioned into smaller sets.
  - We divide by $\boldsymbol{2^{n-15.}}$ This is also the required number of threads.
  - Leave the remaining distribution as-is from the 2-pass round robin algorithm.

Configuration Management

- We conduct a small timing study up to ***n*** = 150 as a proof of concept.
- The max time always occurs at the largest *q*-maximum $max_{(n,1)}$ for any n.
- This small timing study saves a single computation from a single partition from $max_{(n,1)}$ for n = 15, …, 150.
- The next slide shows a graph of the input size versus the run-times.
  - Using multiple nodes and multiple cores.

Configuration Management



Configuration Management

- The graph on the previous slide has 2 plateaus.
  - These plateaus are probability due to the amount of nested loops as ***n*** increases.

- Additional obstacles must be overcome before computing large power sets:
  - Compute the factorial of a number larger than n = 150; say n = 1,000 to 45,136.
  - Automatically monitor the queue; and submit a batch job when one job has finished.

Configuration Management

- Using the model from the timing study y = 0.0008n + 0.0883, it is estimated that it will take 36.1971 seconds to compute the largest partition for the power set for n = 45,136.
  - The remaining tasks are smaller and will take less time.
  - On the Stampede2 SKX compute nodes

Questions

- Thank you for attending.

- Does anyone have any questions?

- Profile and research: https://rogerlgoodwin.brandyourself.com